Fraïsse Theory Notes

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The following are some notes I wrote for a talk I gave for a seminar about Fraïsse theory in the context of descriptive set theory.

If we have some structure $\mathcal{M}$, we can consider the class of finitely generated substructures of $\mathcal{M}$. In Fraïsse's terminology, we call this class the **age** of $\mathcal{M}$, and denote it by $\text{Age}(\mathcal{M})$. One might ask, does $\text{Age}(\mathcal{M})$ contain enough data to determine $\mathcal{M}$. In general, the answer is no. For instance, every infinite linear order has the same age, that being the class of finite linear orders. One goal of Fraïsse theory is to construct one countable structure which is in some way the "cannonical" choice of structure of a given age. Additionally, we seek to find the conditions on a class of finitely generated structures that can tell us if that class is the age of some structure.

Throughout, we will be concerned only with countable structures over a countable first order language $\mathcal{L}$. Also, when considering a class of finitely generated structures K, we will be a little loose about membership to K in so far as we only care about the isomorphism type of the structures in K. That is, we might say that a structure is in $\mathcal{K}$ when we might more precisely mean that a structure is isomorphic to some structure in K. In this vein as well, we shall consider a class K countable if it contains only countably many structures up to isomorphism.

Definition 1. If $\mathcal{M}$ is a structure, and $S \subseteq M$, then we let $\langle S \rangle_{\mathcal{M}}$ denote the substructure of $\mathcal{M}$ generated by S. Then we define

$$\text{Age}(\mathcal{M}) = \{\langle S \rangle_{\mathcal{M}} \mid S \subseteq A \text{ is finite}\}.$$

We start by noticing some properties that $\text{Age}(\mathcal{M})$ will always satsify

Proposition 2. For any structure $\mathcal{M}$ the following two properties are satisfied by $K = \text{Age}(\mathcal{M})$

- The **Hereditary Property** (HP): If $\mathcal{A} \in K$ and $\mathcal{B} \subseteq \mathcal{A}$ is a finitely generated substructure of $\mathcal{A}$, then $\mathcal{B} \in K$.
- The **Joint Embedding Property** (JEP): If $\mathcal{A}, \mathcal{B} \in K$ then there exist some $\mathcal{C} \in K$ such that $\mathcal{A}$ and $\mathcal{B}$ both embed into $\mathcal{C}$.

Proof. For the hereditary property, take an arbitrary $\mathcal{A} = \langle S \rangle_{\mathcal{M}} \in \text{Age}(\mathcal{M})$, and suppose $\mathcal{B} = \langle T \rangle_{\mathcal{A}}$ is an arbitrary finitely generated substructure of $\mathcal{A}$. It is easily verified that

$$\langle T \rangle_{\mathcal{A}} = \langle T \rangle_{\mathcal{M}},$$

Hence, $\mathcal{B} \in \text{Age}(\mathcal{M})$.

For the joint embedding property, suppose we have any two finitely generated substructures $\mathcal{A} = \langle S \rangle_{\mathcal{M}}$ and $\mathcal{B} = \langle T \rangle_{\mathcal{M}}$. Then the structure $\mathcal{C} = \langle S \cup T \rangle_{\mathcal{M}}$ embeds both $\mathcal{A}$ and $\mathcal{B}$ by inclusion maps. □

Our first theorem is in a way a converse to this proposition:

Theorem 3. If K is a countable class of finitely generated structures that satisfies both HP and JEP, then there is a countable structure $\mathcal{M}$ such that $K = \text{Age}(\mathcal{M})$.

Proof. We may list $K = \{\mathcal{S}_n \mid n \in \omega\}$. We will build $\mathcal{M}$ recurisvely as follows. Fix $\mathcal{M}_0$ to be any structure in K. Then, using the joint embedding property, find some $\mathcal{M}_1 \in K$ such that both $\mathcal{M}_0$ and $\mathcal{S}_0$ embed into it, and fix one such embedding $f_0 : \mathcal{M}_0 \to \mathcal{M}_1$. Continue doing this, building a structure $\mathcal{M}_{n+1}$ that emebds both $\mathcal{M}_n$ and $\mathcal{S}_n$ for all n.

$$\begin{array}{ccccccccc} & & \mathcal{S}_0 & & \mathcal{S}_1 & & \mathcal{S}_2 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \mathcal{M}_0 & \xrightarrow{f_0} & \mathcal{M}_1 & \xrightarrow{f_1} & \mathcal{M}_2 & \xrightarrow{f_2} & \mathcal{M}_3 & \cdots & \mathcal{M} \end{array}$$

Now, take $\mathcal{M}$ to be the direct limit of the $\mathcal{M}_n$ with the embeddings f_n. Firstly, each $\mathcal{M}_n$ is finitely generated. Hence, $\mathcal{M}$ will be countably generated and thus countable. Next, it is clear that $K \subseteq \mathrm{Age}(\mathcal{M})$, since for each n we have $\mathcal{S}_n \hookrightarrow \mathcal{M}_{n+1} \hookrightarrow \mathcal{M}$. Finally, suppose $\langle T\rangle_{\mathcal{M}}$ is a finitely generated substructure of $\mathcal{M}$. Since T is finite, there must be some finite n so that $T \subseteq \mathcal{M}_n$. In this case, we have

$$\langle T\rangle_{\mathcal{M}} = \langle T\rangle_{\mathcal{M}_n}.$$

By the hereditary property, it follows that $\langle T\rangle_{\mathcal{M}_n}$ is in K. Thus, we also have $K \supseteq \mathrm{Age}(\mathcal{M})$. □

This theorem together with the preceding propositon give us a precise characterization of which classes of finitely generated structures are the Age of some structure. As noted previously however, this theorem has no guarantees about the canonicity of the structure $\mathcal{M}$ we build from K. One hope is that if we can show that there is one particular structure that shows up "most of the time" when we do this construction, then we can confidently point to that structure as being the right structure. To take this approach, we can rigorize this idea of "showing up most of the time" with the notion of comeagerness in an appropriate Polish space. With this motivation, we define the following Polish space associated to a first order language $\mathcal{L}$:

Definition 4. Let $\mathcal{L}$ be a countable, first order language. Define the space

$$X_{\mathcal{L}} = \prod_{c\in\mathcal{L}} \omega \times \prod_{R\in\mathcal{L}} 2^{(\omega^{n(R)})} \times \prod_{f\in\mathcal{L}} \omega^{(\omega^{n(f)})},$$

taken with the product topology, where the products range over the constants, relations, and functions in $\mathcal{L}$ respectively, and where $n(R)$ or $n(f)$ denotes the arity of a relation or function symbol. We call $\mathcal{L}$ the space of $\mathcal{L}$ structures with domain ω, since each element of $X_{\mathcal{L}}$ specifies exactly the data of the interpretations of the symbols of $\mathcal{L}$ onto ω.

We observe that if there are any relation symbols in the language, the middle factor always looks likea copy of Cantor space 2^ω. Similarly, if there are any function symbols in $\mathcal{L}$ then the last factor always looks like a copy of Baire space ω^ω. If there is a positive but finite number of constants in $\mathcal{L}$, then the first term looks like a discrete space ω, and if there are infinitely many constants then it also looks like Baire space. In summary, $X_{\mathcal{L}}$ will look like a product of some subset of $\{\omega, 2^\omega, \omega^\omega\}$. Of particular note is that if $\mathcal{L}$ is completely relational, then $X_{\mathcal{L}}$ will be isomorphic to Cantor space, which has the special property of being compact.

To interface with this Polish space, we may use a particularly convenient countable basis. If $\varphi(\overline{x})$ is a quantifier free formula in $\mathcal{L}$, and $\overline{a}$ is a tuple in ω, then we can define the following clopen set

$$N_{\phi,\overline{a}} = \{\mathcal{M} \in X_{\mathcal{L}} \mid \mathcal{M} \models \varphi[\overline{a}]\}.$$

In fact, it can be checked that these sets form a countable basis for the Polish toplogy on $X_{\mathcal{L}}$.

We can now return to our analysis of the age of countable structures. For our purposes, $X_{\mathcal{L}}$ is still much too large. We only care about the behavior of a structure amongst all of the other structures with the same age. To this end, we define the following subspace of $X_{\mathcal{L}}$.

Definition 5. If K is a class of structures, let

$$X_K = \{\mathcal{M} \in X_{\mathcal{L}} \mid \text{Age}(\mathcal{M}) = K\}.$$

Proposition 6. If $\mathcal{L}$ is finite and K is a countable class of finite structures then X_K is G_δ in $X_{\mathcal{L}}$. Hence, X_K is a Polish space with the subspace topology from $X_{\mathcal{L}}$

Proof. Suppose $\mathcal{L}$ is finite and $\mathcal{A}$ is a finite $\mathcal{L}$-structure with domain $\{a_1, \dots, a_n\}$. Then there exists a single quantifier free, first order formula $\varphi(x_1, \dots, x_n)$ such that for any other structure $\mathcal{M}$ and elements $m_1, \dots, m_n \in M$, we have

$$\mathcal{M} \models \varphi(m_1, \dots, m_n) \Leftrightarrow a_i \mapsto a_n \text{ is an isomorphism onto its image.}$$

This quantifier free formula simply consists of taking the conjunction of all statements in the atomic diagram of the structure, which will be finite given that $\mathcal{A}$ and $\mathcal{L}$ are finite. Given this formula, it is not hard to see that

$$\begin{aligned} \{\mathcal{M} \in X_L \mid \mathcal{A} \text{ embeds into } \mathcal{M}\} &= \{\mathcal{M} \models \exists m_1, \dots, m_n \in \omega[\mathcal{M} \models \varphi(m_1, \dots, m_n)].\} \\ &= \bigcup_{\overline{m} \in \omega^n} N_{\varphi, \overline{m}} \end{aligned}$$

As this union is the countable union of basic clopen sets in X_L, this shows that this set is open. Now observe that

$$\{\mathcal{M} \in X_{\mathcal{L}} \mid K \subseteq \text{Age}(\mathcal{M})\} = \bigcap_{\mathcal{A} \in K} \{\mathcal{M} \in X_L \mid \mathcal{A} \text{ embeds into } \mathcal{M}\}.$$

Hence, this set is G_δ. On the other hand, suppose we fix such a formula $\varphi_{\mathcal{A}}$ for each $\mathcal{A} \in K$. Then we have

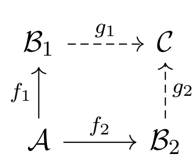
$$\{\mathcal{M} \in X_{\mathcal{L}} \mid \text{Age}(\mathcal{M}) \subseteq K\} = \bigcap_{\overline{m} \in \omega^{<\omega}} \bigcup_{\substack{\mathcal{A} \in K \\ |\mathcal{A}| = |\overline{m}|}} N_{\varphi_A, \overline{m}}$$

is also G_δ. The intersection of these two sets is X_K, thus X_K is G_δ. ☐

As an aside, note that the key work being done in this proof is that for finite structures $\mathcal{A}$, there is a existential sentence φ so that a structure embeds $\mathcal{A}$ if and only if it satisfies φ. This statement is true about a broader class of structures, and this leads to ideas involving the notion of a projectively isolated type.

With this space now defined, we may once again return to our original focus. In view of Theorem 3, we know that if K satisfies HP and JEP, then X_K is nonempty. We may now ask, is there a single structure in X_K whose isomorphism class is comeager? The answer is yes, given that the following property of K also holds

Definition 7. The **Amalgamation Property** holds of K if for any structures $\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2 \in K$, along with embeddings $f_i : \mathcal{A} \to \mathcal{B}_i$, there is a structure $\mathcal{C} \in K$ (called the amalgam of $\mathcal{B}_1, \mathcal{B}_2$) and embeddings $g_i : \mathcal{B}_i \to \mathcal{C}$ such that $g_1 \circ f_1 = g_2 \circ f_2$. This can be summarized by the following picture:



While this might seem like a strict generalization of the JEP, this is not the case if K does not have a common initial structure. For instance, if K is the class of finite fields, in fact K does have the amalgamation property. However, there is no joint embedding of two finite fields of different characteristic.

If K is a countable class of finite structures satisfying HP, JEP, and AP, then we call K a **Fraïsse** class. As we will show, if K is a Fraïsse class, then X_K has a comeager isomorphism class. To prove this, we show that a certain property called *homogeneity* is comeager, and that any two homogeneous structures are isomorphic.

Definition 8. A countable structure $\mathcal{M}$ is **homogenous** if any isomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ between finite substructures of $\mathcal{M}$ extends to an automorphism of $\mathcal{M}$.

From now on, we will call an isomorphism of finite substructures of $\mathcal{M}$ a finite partial automorphism. In practice, it is often much easier to verify a seamingly weaker sufficient condition for homogeneity, often called weak homogeneity.

Lemma 9. Suppose $\mathcal{M}$ is a countable structure such that for any finite partial automorphism φ of $\mathcal{M}$, and any element $a \in \mathcal{M}$, there is a finite partial automorphism $\overline{\varphi}$ extending φ such that $a \in \operatorname{dom}(\overline{\varphi})$. Then $\mathcal{M}$ is homogeneous.

Proof. Suppose $\varphi : \mathcal{A} \to \mathcal{B}$ is a finite partial automorphism of $\mathcal{M}$. We will use the hypothesis of the lemma to extend φ one element at a time to be a full automorphism of $\mathcal{M}$. Enumerate the elements of $\mathcal{M}$ as $(a_n)_{n\in\omega}$. Let $f_0 = \varphi$. Inductively, we construct f_n for $n \in \omega$, and let $f = \bigcup_{n\in\omega} f_n$. If $n = 2k+1$, we apply the hypothesis of the lemma to f_{n-1} and a_k to ensure $a_k \in \operatorname{dom}(f_n)$. If $n = 2k+2$ we apply the hypothesis of the lemma to f_{n-1}^{-1} and k to ensure $a_k \in \operatorname{ran}(f_n)$. Hence, in the limit, for all k we have $a_k \in \operatorname{dom}(f)$ and $a_k \in \operatorname{ran}(f)$. Thus, f is an automorphism of $\mathcal{M}$. □

Lemma 10. If two countable structures are both homogeneous and have the same age, then they are isomorphic.

Proof. Take two countable homogeneous structures $\mathcal{M}$ and $\mathcal{N}$ with the same age $K = \operatorname{Age}(\mathcal{M}) = \mathcal{N}$. List out the elements of $\mathcal{M}$ as $\{a_i\}_{i\in\omega}$ and the elements of $\mathcal{N}$ as $\{b_i\}_{i\in\omega}$. We build an isomorphism between $\mathcal{M}$ and $\mathcal{N}$ through a back and forth argument. We will inductively build an increasing ladder of finitely generated substructures $\mathcal{A}_n \subseteq \mathcal{M}$ and $\mathcal{B}_n \subseteq \mathcal{N}$ with isomorphisms $f_n : \mathcal{A}_n \to \mathcal{B}_n$ that extend each other.

$$\begin{array}{ccccccc} \mathcal{A}_0 & \subseteq & \mathcal{A}_1 & \subseteq & \mathcal{A}_2 & \cdots \subseteq & \mathcal{M} \\ \big\downarrow{\scriptstyle f_0} & & \big\downarrow{\scriptstyle f_1} & & \big\downarrow{\scriptstyle f_2} & & \\ \mathcal{B}_0 & \subseteq & \mathcal{B}_1 & \subseteq & \mathcal{B}_2 & \cdots \subseteq & \mathcal{N} \end{array}$$

Fix some structure $\mathcal{S} \in K$, and fix copies $\mathcal{A}_0 \subseteq \mathcal{M}$ and $\mathcal{B}_0 \subseteq \mathcal{N}$ which are both isomorphic to $\mathcal{S}$. Fix an isomorphism $f_0 : \mathcal{A}_0 \to \mathcal{B}_0$. Next, on an odd stage $n = 2k+1$, we ensure that $a_k \in \operatorname{dom}(f_n)$. Let $\mathcal{S}$ be the substructure of $\mathcal{M}$ generated by $\mathcal{A}_{n-1}$ and a_k. Since $\mathcal{S}$, is a finitely generated substructure of $\mathcal{M}$, we have $\mathcal{S} \in \operatorname{Age}(\mathcal{M}) = \operatorname{Age}(\mathcal{N})$. Hence, there is some $\mathcal{B}' \subseteq \mathcal{N}$ with $\mathcal{B}' \cong \mathcal{S}$. Let g be a particular isomorphism $\mathcal{S} \to \mathcal{B}'$. We have the following picture:

$$\begin{array}{ccccc} & & a_k & & \\ & & \Cap & & \\ \mathcal{A}_{n-1} & \subseteq & \mathcal{S} & \subseteq & \mathcal{M} \\ \big\downarrow{\scriptstyle f_{n-1}} & & \big\downarrow{\scriptstyle g} & & \\ \mathcal{B}_{n-1} & & \mathcal{B}' & \subseteq & \mathcal{N} \end{array}$$

Ideally, we would want $\mathcal{B}'$ to extend $\mathcal{B}_{n-1}$, so that we can continue the picture. However, all we know is that $\mathcal{S}$ lives as a copy somewhere in $\mathcal{N}$. This is where we can use the homogeneity of $\mathcal{N}$ to essentially move this copy of $\mathcal{S}$ back over to extend $\mathcal{B}_{n-1}$ as desired.

Notice in this diagram that we have two copies of $\mathcal{B}_{n-1}$ living inside $\mathcal{N}$, with the isomorphism $g \circ f_{n-1}^{-1}$ going between them. Since $\mathcal{N}$ is homogeneous, $g \circ f_{n-1}^{-1}$ extends to an automorphism h of $\mathcal{N}$. Now, $h^{-1}[\mathcal{B}'] \supseteq \mathcal{B}_{n-1}$. Moreover, $\mathcal{S}$ is isomorphic to $h^{-1}[\mathcal{B}']$ via an isomorphism that extends f_{n-1}, namely $h^-1 \circ g$. Hence, we may let $\mathcal{A}_n = \mathcal{S}$ and $\mathcal{B}_n = h^{-1}[\mathcal{B}']$.

On an even step $n = 2k+2$, we may do a very similar thing to ensure that $b_k \in \text{ran}(f_n)$ by flipping the diagram and working with f_i^{-1}. This completes the ladder. Since we ensured that $a_k \in \mathcal{A}_{2k+1}$ and $b_k \in \mathcal{B}_{2k+2}$, we have that $\cup_n \mathcal{A}_n = \mathcal{M}$ and $\cup_n \mathcal{B}_n = \mathcal{N}$. Moreover, taking the limit of the f_n yields an isomorphism $\mathcal{M} \cong \mathcal{N}$. □

Theorem 11. If $\mathcal{L}$ is finite and K is Fraïsse class, then the homogenous structures in X_K form a comeager isomorphism class.

Proof. From lemma 10, it suffices to show that the property of being homogeneous is comeager, as apriori all structures in X_K have the same age. In fact, we will show that homogeneity is dense and G_δ. We use the sufficient condition described in lemma 9 to show that it is G_δ. We can quantify over all partial isomorphisms of $\mathcal{M}$ as pairs of lists $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ such that the map $a_i \mapsto b_i$ induces an isomorphism $\langle \overline{a} \rangle_{\mathcal{M}} \cong \langle \overline{b} \rangle_{\mathcal{M}}$. As we noted in the proof of Proposition 6, if $\langle \overline{a} \rangle_{\mathcal{M}}$ is finite, we can find a quantifier free formula that captures the notion of being an isomorphic to $\langle \overline{a} \rangle_{\mathcal{M}}$. Specifically, there is a quantifier free formula $\varphi_{\langle \overline{a} \rangle_{\mathcal{M}}}(\overline{x})$ such that $\mathcal{N} \models \varphi_{\overline{a}}(\overline{b})$ if and only if $a_i \mapsto b_i$ induces an isomorphism $\langle \overline{a} \rangle_{\mathcal{M}} \cong \langle \overline{b} \rangle_{\mathcal{N}}$. Hence, we have that

$$\mathcal{M} \text{ is homogeneous } \Leftrightarrow \forall \overline{a}, a', \overline{b} \in \omega \; [\mathcal{M} \models \varphi_{\langle \overline{a} \rangle_{\mathcal{M}}}(\overline{b}) \to \exists b' \in \omega \; [\mathcal{M} \models \varphi_{\langle \overline{a}, a' \rangle_{\mathcal{M}}}(\overline{b}, b')]].$$

This shows that being homoegenous is a G_δ property in X_K.

Now, we show it is dense. From the above, we know that the set of homogeneous $\mathcal{M}$ is the intersection of the sets

$$W_{\overline{a}, a', \overline{b}} = \{\mathcal{M} \in X_K \mid \mathcal{M} \models \varphi_{\langle \overline{a} \rangle_{\mathcal{M}}}(\overline{b}) \to \exists b' \in \omega \; [\mathcal{M} \models \varphi_{\langle \overline{a}, a' \rangle_{\mathcal{M}}}(\overline{b}, b')]\}$$

across all $\overline{a}, a', \overline{b}$. Since each $W_{\overline{a}, a', \overline{b}}$ is open, to show that the intersection of these $W_{\overline{a}, a', \overline{b}}$ is dense, it suffices to show each one is dense by the Baire Category theorem.

Fix a set $W_{\overline{a}, a', \overline{b}}$, and fix a nonempty, basic open set $N_{\psi, \overline{c}}$. It remains to show that $W_{\overline{a}, a', \overline{b}} \cap N_{\psi, \overline{c}}$ is nonempty. As $N_{\psi, \overline{c}}$ is nonempty, that means there is some $\mathcal{N} \in X_K$ with $\mathcal{N} \models \psi(\overline{c})$. If $\mathcal{N} \not\models \varphi_{\langle \overline{a} \rangle_{\mathcal{M}}}(\overline{b})$ then trivially $\mathcal{N} \in W_{\overline{a}, a', \overline{b}}$. Hence, $W_{\overline{a}, a', \overline{b}} \cap N_{\psi, \overline{c}} \neq \emptyset$. Otherwise, consider the following amalgamation diagram:

$$\begin{array}{ccc} \langle \overline{a}, a' \rangle_{\mathcal{N}} & \overset{f}{\dashrightarrow} & \mathcal{S} \\ \cup\!\! \mid & & \uparrow g \\ \langle \overline{a} \rangle_{\mathcal{N}} & \xrightarrow{a_i \mapsto b_i} & \langle \overline{b}, \overline{a}, a', \overline{c} \rangle_{\mathcal{N}} \end{array}$$

Since $\mathcal{N} \in X_K$, we have $\text{Age}(\mathcal{N}) = K$. In particular, the three structures in the bottom and left of the diagram are all in K. Moreover, since $\mathcal{N} \models \varphi_{\langle \overline{a} \rangle_{\mathcal{M}}}$, it follows that the map on the bottom of the diagram is an embedding of structures. Then using the amalgamation property of K, we find that there exists a structure $\mathcal{S} \in K$ and embeddings f, g into $\mathcal{S}$ making the diagram commute. The idea is that $\mathcal{S}$ captures the information needed to ensure that a structure ends up in $W_{\overline{a}, a', \overline{b}} \cap N_{\psi, \overline{c}}$.

Take some countable structure $\tilde{\mathcal{M}}$ with $\mathrm{Age}(\tilde{\mathcal{M}}) = K$. We will not yet declare that $\tilde{\mathcal{M}}$ has domain ω, as we will want to choose a specific labeling of its elements to make a structure in $W_{\overline{a},a',\overline{b}} \cap N_{\psi,\overline{c}}$. Since $\mathrm{Age}(\tilde{\mathcal{M}}) = K$, there is some embedding $h : \mathcal{S} \to \tilde{\mathcal{M}}$. From this, we get the composite embedding $h \circ g : \langle \overline{b}, \overline{a}, a', \overline{c} \rangle_{\mathcal{N}} \to \tilde{\mathcal{M}}$. Only now do we label the elements of $\tilde{\mathcal{M}}$ with naturals in such a way that the embedding $h \circ g$ becomes the identity map. In other words, if we call this labeled structure $\mathcal{M}$, then we properly have that $\langle \overline{b}, \overline{a}, a', \overline{c} \rangle_{\mathcal{N}}$ is a substructure of $\mathcal{M}$. It follows that $\langle \overline{b}, \overline{a}, a', \overline{c} \rangle_{\mathcal{N}} = \langle \overline{b}, \overline{a}, a', \overline{c} \rangle_{\mathcal{M}}$. Additionally, we may identify $\mathcal{S}$ with $h[\mathcal{S}] \subset \mathcal{M}$, so that we now have the picture:

$$\begin{array}{ccc} & & \mathcal{M} \\ & & \cup\!\!\mid \\ \langle \overline{a}, a' \rangle_{\mathcal{M}} & \xrightarrow{\quad f \quad} & \mathcal{S} \\ \cup\!\!\mid & & \cup\!\!\mid \\ \langle \overline{a} \rangle_{\mathcal{M}} & \xrightarrow{a_i \mapsto b_i} & \langle \overline{b}, \overline{a}, a', \overline{c} \rangle_{\mathcal{M}} \end{array} \quad .$$

Firstly, it clearly follows from $\langle \overline{b}, \overline{a}, a', \overline{c} \rangle_{\mathcal{N}} = \langle \overline{b}, \overline{a}, a', \overline{c} \rangle_{\mathcal{M}}$ that $M \models \psi(\overline{c})$. Hence, $\mathcal{M} \in N_{\psi,\overline{c}}$. Secondly, consider $b' = f(a')$. Since f is an embedding and the diagram commutes, we have $\langle \overline{a}, a' \rangle_{\mathcal{M}} \cong \langle \overline{b}, b' \rangle$ by sending $a_i \mapsto b_i$ and $a' \mapsto b'$. Thus, we also have that $\mathcal{M} \in W_{\overline{a},a',\overline{b}}$. $\square$