Notation 1. Throughout this exposition, $\mathcal{L}$ will denote a first order language, T will denote an $\mathcal{L}$-theory, $\mathcal{M}$ will denote an $\mathcal{L}$ structure with universe M, and $\varphi(\overline{x}, \overline{y})$ will be an $\mathcal{L}$-formula in $n+m$ free variables, where $|\overline{x}| = n$ and $|\overline{y}| = m$. For any ordinal α (including natural numbers), let 2^α denote the set of binary strings of length α, i.e. functions from α to the set $2 = \{0, 1\}$, and let $2^{<\alpha}$ denote the set of binary strings of length less than α. In this context, $2^{<\alpha}$ will be called a tree, ordered under $\preceq$, and an element of 2^α might be called a branch.

Definition 2. For an ordinal α, we say $\Gamma_T(\varphi, \alpha)$ holds if there exists a model $\mathcal{M} \models T$, and a height-α binary tree of m-tuples in M, i.e. a function $f : 2^{<\alpha} \to M^m$, such that for any branch in the tree $\tau \in 2^\alpha$, the collection of formulas

$$\Sigma_\tau = \{\varphi(\overline{x}, f(\tau\restriction\beta)) \mid \beta < \alpha, \tau(\beta) = 0\} \cup \{\neg\varphi(\overline{x}, f(\tau\restriction\beta)) \mid \beta < \alpha, \tau(\beta) = 1\}$$

is satisfiable in $\mathcal{M}$.

Proposition 3. If $n \in \mathbb{N}$, then there is a finite $\mathcal{L}$-formula that says "there is a witness to $\Gamma_T(\varphi, n)$", which I will denote by $\gamma(\varphi, n)$. This can be seen as the formula

$$\exists_{\sigma \in 2^{<n}} \overline{y}_\sigma \bigwedge_{\tau \in 2^n} \exists \overline{x} \bigwedge_{k<n} \varphi(\overline{x}, \overline{y}_{\tau\restriction k}) \leftrightarrow \tau(k) = 0,$$

where $\leftrightarrow \tau(k) = 0$ is shorthand for either the formula or its negation depending on whether $\tau(k) = 0$.

Definition 4. Let I be a linear order. We say that $B_T(\varphi, I)$ holds if there exists a model $\mathcal{M} \models T$, and sequences $\langle \overline{a}_i \mid i \in I\rangle$ in M^n and $\langle \overline{b}_i \mid i \in I\rangle$ in M^m such that for all $i, j \in I$,

$$\mathcal{M} \models \varphi(\overline{a}_i, \overline{b}_j) \Leftrightarrow i < j.$$

Lemma 5. If for all $k \in \mathbb{N}$, treating k as the linear order with k elements, $B_T(\varphi, k)$ holds, then $B_T(\varphi, I)$ holds for all linear orders I.

Proof. Let I be an arbitrary linear order. Define the language

$$\mathcal{L}' = \mathcal{L} \cup \{c_{i,k} \mid i \in I, k < n\} \cup \{d_{i,k} \mid i \in I, k < m\},$$

an expansion by constants. For notation's sake, let $\overline{c}_i$ denote the tuple $c_{i,0}, \ldots, c_{i,n-1}$ and let $\overline{d}_i$ denote the tuple $d_{i,0}, \ldots, d_{i,m-1}$. Let T' be the theory

$$T \cup \{\varphi(\overline{c}_i, \overline{d}_j) \mid i, j \in I, i < j\} \cup \{\neg\varphi(\overline{c}_i, \overline{d}_j) \mid i, j \in I, i \geq j\}.$$

We will show that T' is consistent. Let Σ be a finite subset of T'. For some finite $I' \subseteq I$, we have

$$\Sigma \subseteq T \cup \{\varphi(\overline{c}_i, \overline{d}_j) \mid i, j \in I', i < j\} \cup \{\neg\varphi(\overline{c}_i, \overline{d}_j) \mid i, j \in I', i \geq j\}.$$

We can enumerate I' as $i_1 < i_2 < \cdots < i_\ell$. By assumption, there is a model $\mathcal{M} \models T$ and sequences $\overline{a}_1, \ldots, \overline{a}_\ell$ and $\overline{b}_1, \ldots, \overline{b}_\ell$ such that for all $1 \leq k, l \leq \ell$,

$$\mathcal{M} \models \varphi(a_k, b_\ell) \Leftrightarrow k < \ell.$$

Consider the $\mathcal{L}'$-structure $\mathcal{M}'$ built from $\mathcal{M}$ by realizing $\overline{c}_{i_k}, \overline{d}_{i_k}$ for $1 \leq k \leq \ell$ as $\overline{a}_k$ and $\overline{b}_k$ respectively, and for $i \in I \setminus I'$, realizing $\overline{c}_i, \overline{d}_i$ as $\overline{a}_0$ and $\overline{b}_0$ respectively.

Let $\theta \in \Sigma$. If $\theta \in T$, then $\mathcal{M}' \models \theta$ since $\mathcal{M} \models \theta$, and $\mathcal{M}'$ is the same as $\mathcal{M}$ on $\mathcal{L}$ symbols. Otherwise, $\theta \in \{\varphi(\overline{c}_i, \overline{d}_j) \mid i, j \in I', i < j\} \cup \{\neg\varphi(\overline{c}_i, \overline{d}_j) \mid i, j \in I', i \geq j\}$. Then $\mathcal{M}' \models \theta$ since

$$\mathcal{M}' \models \varphi(\overline{c}_{i_k}, \overline{d}_{i_{k'}}) \Leftrightarrow \mathcal{M} \models \varphi(\overline{a}_k, \overline{b}_{k'}) \Leftrightarrow k < k' \Leftrightarrow i_k < i_{k'}.$$

Thus, Σ is satisfiable and so T' is consistent. $\square$

Lemma 6. Any finite subset of a tree is contained in a finite subtree of that tree.

Lemma 7. If $\Gamma_T(\varphi, k)$ holds for all $k \in \mathbb{N}$, then $\Gamma_T(\varphi, \alpha)$ holds for all ordinals α.

Proof. Suppose $\Gamma_T(\varphi, k)$ holds for all $k \in \mathbb{N}$. Let α be any ordinal. Define the language

$$\mathcal{L}' = \mathcal{L} \cup \{d_{\sigma,k} \mid \sigma \in 2^{<\alpha}, k < m\} \cup \{c_{\tau,k} \mid \tau \in 2^{\alpha}, k < n\},$$

an expansion by constants. For notation's sake, let $\overline{d}_\sigma$ denote the tuple $d_{\sigma,0}, \ldots, d_{\sigma,m-1}$ and let $\overline{c}_\tau$ denote the tuple $c_{\tau,0}, \ldots, c_{\tau,n-1}$. Let T' be the theory

$$T \cup \{\varphi(\overline{c}_\tau, \overline{d}_{\tau\restriction\beta}) \mid \tau \in 2^{\alpha}, \tau(\beta) = 0\} \cup \{\neg\varphi(\overline{c}_\tau, \overline{d}_{\tau\restriction\beta}) \mid \tau \in 2^{\alpha}, \tau(\beta) = 1\}.$$

If we show that T' has a model $\mathcal{M}$, then $\Gamma_T(\varphi, \alpha)$ holds, as witnessed by the tree $\sigma \mapsto \overline{c}_\sigma^{\mathcal{M}}$. Thus, it suffices to show that T' is consistent. By compactness, it sufficies to show that every finite subset of T' is consistent, so let Σ be a finite subset of T'. For some finite $X \subseteq 2^{\alpha}$ and $A \subseteq \alpha$ we have

$$\Sigma \subseteq T \cup \{\varphi(\overline{c}_\tau, \overline{d}_{\tau\restriction\beta}) \mid \tau \in X, \tau(\beta) = 0, \beta \in A\} \cup \{\neg\varphi(\overline{c}_\tau, \overline{d}_{\tau\restriction\beta}) \mid \tau \in 2^{\alpha}, \tau(\beta) = 1, \beta \in A\}.$$

Let $Y = \{\tau\restriction\beta \mid \beta \in A\}$. By lemma 6, Y is contained in a subtree of $2^{<\alpha}$. That means there is a map $g : 2^{<\ell} \to 2^{<\alpha}$ such that Y is contained in the image of g. Choose $\mathcal{M} \models T$ and a tree $f : 2^{<\ell} \to M^m$ as a witness to $\Gamma_T(\varphi, \ell)$.

We will now build a model $\mathcal{M}' \models T'$ starting with $\mathcal{M}$. For any $\sigma \in Y$, intepret $\overline{d}_\sigma$ in $\mathcal{M}'$ as $f(\rho)$, where $\rho \in 2^{<\ell}$ such that $g(\rho) = \sigma$. Next, let $\tau \in X$. Consider $\rho' = \cup\{\sigma \in 2^{<\ell} \mid g(\sigma) \prec \tau\}$. Choose some $\rho \in 2^{\ell}$ extending ρ'. Since f witnesses $\Gamma_T(\varphi, \ell)$, we have

$$\Delta = \{\varphi(\overline{x}, f(\rho\restriction k)) \mid k < \ell, \rho(k) = 0\} \cup \{\neg\varphi(\overline{x}, f(\rho\restriction k)) \mid k < \ell, \rho(k) = 1\}$$

is satisfiable in $\mathcal{M}$. Then interpret $\overline{c}_\tau$ in $\mathcal{M}'$ as a witness to Δ. Interpret all other constants arbitrarily. If $\beta \in A$, then there is a $\sigma \in 2^{<\ell}$ such that $\tau\restriction\beta = g(\sigma)$. We have interpreted $\overline{c}_\tau$ such that

$$\mathcal{M}' \models \varphi(\overline{c}_\tau, f(\rho\restriction|\sigma|)) \Leftrightarrow \tau(\beta) = 0.$$

Furthermore, $f(\rho\restriction|\sigma|) = f(\sigma) = d_{g(\sigma)}^{\mathcal{M}'} = d_{\tau\restriction\beta}^{\mathcal{M}'}$, so we have

$$\mathcal{M}' \models \varphi(\overline{c}_\tau, d_{\tau\restriction\beta}) \Leftrightarrow \tau(\beta) = 0.$$

Thus, $\mathcal{M}' \models \Sigma$.

□

Theorem 8 (INCOMPLETE)**.** For any formula $\varphi(\overline{x}, \overline{y})$, the following are equivalent

(1) $B_T(\varphi, \mathbb{N})$ holds.

(2) $B_T(\varphi, I)$ holds for all orders I.

(3) For all α an ordinal, $\Gamma_T(\varphi, \alpha)$ holds.

(4) $\Gamma_T(\varphi, \omega)$ holds.

Proof. We will show (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2). Suppose $B_T(\varphi, \mathbb{N})$ holds. Then there exists a model $\mathcal{M} \models T$ and sequences $\langle \overline{a}_i \mid i \in \mathbb{N}\rangle$ and $\langle \overline{b}_i \mid i \in \mathbb{N}\rangle$ such that for all $i, j \in \mathbb{N}$,

$$\mathcal{M} \models \varphi(\overline{a}_i, \overline{b}_j) \Leftrightarrow i < j.$$

By lemma 5, it is sufficient to show that $B_T(\varphi, k)$ holds for all $k \in \mathbb{N}$. Clearly $\overline{a}_0, \ldots, \overline{a}_{k-1}$ and $\overline{a}_0, \ldots, \overline{a}_{k-1}$ witness this.

(2) $\Rightarrow$ (3). Clearly if $\beta < \alpha$ and $\Gamma_T(\varphi, \alpha)$ then $\Gamma_T(\varphi, \beta)$. Thus, it is sufficient to show the result for all limit ordinals, since any ordinal is less than some limit. Let α be any limit ordinal. Let $I = 2^\alpha$, the set of functions $\alpha \to 2$. Order I lexicographically, that is $\sigma < \tau$ iff $\sigma(\gamma) < \tau(\gamma)$ where γ is the least ordinal where $\sigma(\gamma) \neq \tau(\gamma)$. Equivalently, $\sigma < \tau$ if for some γ, we have $\sigma \upharpoonright \gamma = \tau \upharpoonright \gamma$ and $\sigma(\gamma) < \tau(\gamma)$.

By assumption, $B_T(\varphi, I)$ holds, so let $\mathcal{M} \models T$ and choose $\langle \overline{a}_i \mid i \in I\rangle$ and $\langle \overline{b}_i \mid i \in I\rangle$ from M such that

$$\mathcal{M} \models \varphi(\overline{a}_i, \overline{b}_j) \Leftrightarrow i < j.$$

Define $\iota : 2^{<\alpha} \to I$ by $\sigma \mapsto \sigma^\frown 1000\ldots$ and define the tree $F : 2^{<\alpha} \to M$ by $\sigma \mapsto b_{\iota(\sigma)}$. We will show that this tree is a witness to $\Gamma_T(\varphi, \alpha)$. In particular, if $\sigma : \alpha \to 2$ is a branch through F, then we claim that for all $\beta < \alpha$,

$$\mathcal{M} \models \varphi(a_\sigma, F(\sigma \upharpoonright \beta)) \Leftrightarrow \sigma(\beta) = 0.$$

Let $\beta < \alpha$. First suppose $\sigma(\beta) = 0$. We have $\sigma \upharpoonright \beta = \iota(\sigma \upharpoonright \beta) \upharpoonright \beta$ and $\sigma(\beta) = 0 < 1 = \iota(\sigma \upharpoonright \beta)(\beta)$. Thus, $\sigma < \iota(\sigma \upharpoonright \beta)$. By construction, that means

$$\mathcal{M} \models \varphi(a_\sigma, b_{\iota(\sigma \upharpoonright \beta)}),$$

so $\mathcal{M} \models \varphi(a_\sigma, F(\sigma \upharpoonright \beta))$.

Next suppose $\sigma(\beta) = 1$. Then we have $\sigma \upharpoonright (\beta + 1) = \iota(\sigma \upharpoonright \beta) \upharpoonright (\beta + 1)$. Furthermore, for all $\gamma \geq \beta + 1$, we have $\iota(\sigma \upharpoonright \beta)(\gamma) = 0$. In particular, if ever $\iota(\sigma \upharpoonright \beta)(\gamma) \neq \sigma(\gamma)$ then $\iota(\sigma \upharpoonright \beta)(\gamma) < \sigma(\gamma)$, and so $\iota(\sigma \upharpoonright \beta)(\gamma) \leq \sigma(\gamma)$. Thus, by construction,

$$\mathcal{M} \not\models \varphi(a_\sigma, b_{\iota(\sigma \upharpoonright \beta)}),$$

so $\mathcal{M} \models \neg\varphi(a_\sigma, F(\sigma \upharpoonright \beta))$. Thus, the claim is shown.

(3) $\Rightarrow$ (4) obvious.

(4) $\Rightarrow$ (1) By lemma 5, it is sufficient to show that $B_T(\varphi, n)$ holds for all $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. Let $h_r = 2^{n-r+1} - 2$. Notice that we have the recursive relation $h_{r+1} = h_r/2 - 1$. We will prove by induction on r that there exists an r-ladder $a_1, \ldots, a_r$ and $b_1, \ldots, b_r$, a height h_r tree H, a set of branches $\{a_\sigma \mid \sigma \in 2^{h_r}\}$ and an index $1 \leq q \leq r$. INCOMPLETE

$\square$

Theorem 9. The following are equivalent:

(1) T is unstable.

(2) There is an undefinable type over T.

(3) There is a formula $\varphi(\overline{x}, \overline{y})$ satisfying $\Gamma_T(\varphi, \omega)$

(4) There is an unstable formula φ over T.

Proof. (1) $\Rightarrow$ (2):

Suppose (2) is false. That is, all types are definable over T. Let $\lambda = |\mathcal{L}|$. We will show that T is 2^λ-stable. Let $\mathcal{M} \models T$ and let $A \subseteq M$ satisfy $|A| = 2^\lambda$. Since all types are definable, there is a distinct definition schema for each type in $S^{\mathcal{M}}(A)$, we have $|S^{\mathcal{M}}(A)|$ is at most the number of definition schemata over A. Each definition schema is a function from $\mathcal{L}$-formulas to $\mathcal{L}_A$-formulas. Thus,

$$|S^{\mathcal{M}}(A)| \leq |\mathrm{Form}(\mathcal{L}_A)|^{|\mathrm{Form}(\mathcal{L})|} = (|\mathcal{L}| + |A|)^{|\mathcal{L}|} = (\lambda + 2^\lambda)^\lambda = 2^{\lambda \times \lambda} = 2^\lambda.$$

Thus, T is 2^λ-stable, and so (1) is also false.

(2) $\Rightarrow$ (3):

Assume (3) is false. That is, $\Gamma_T(\varphi, \omega)$ does not hold for any φ. By lemma 7, we have that for all φ, there is a finite k such that $\Gamma_T(\varphi, k)$ does not hold.

Let $\mathcal{M} \models T$, let $A \subseteq M$, and let p be an arbitrary n-type over A. Our goal is to show that p is definable. For any $\varphi(\overline{x}, \overline{y})$ and any $\mathcal{L}_A$-formula θ in n free variables, $\mathcal{M} \models \gamma(\varphi, \theta, k) \Rightarrow \mathcal{M} \models \gamma(\varphi, k)$[1]. Thus, for each θ there is a k such that $\mathcal{M} \not\models \gamma(\varphi, \theta, k)$. Let k be the least value for which there is a $\psi \in p$ such that $\mathcal{M} \not\models \gamma(\varphi, \psi, k)$, and fix some $\psi \in p$ witnessing this. Since p is a consistent type, and $\psi \in p$, there must be some $\overline{a} \in M^n$ with $\mathcal{M} \models \psi(\overline{a})$. In particular, $k > 0$.

For each $\overline{b} \in M^m$, let $\theta_{\overline{b}}(\overline{x}) = \psi(\overline{x}) \wedge \varphi(\overline{x}, \overline{b})$. Note that θ can be thought of as a single formula in $n + m$ free variables. Define $d_p\varphi(\overline{y}) = \gamma(\varphi, \theta_{\overline{y}}, k-1)$. We will show that d_p is a definition for p.

Let $\overline{b} \in A^m$. First suppose $\varphi(\overline{x}, \overline{b}) \in p$. Since both $\psi(\overline{x}) \in p$ and $\varphi(\overline{x}, \overline{b}) \in p$, we have $\theta_{\overline{b}}(\overline{x}) \in p$. By the minimality of k, we have $\mathcal{M} \models \gamma(\varphi, \theta_{\overline{b}}, k-1)$. Thus, $\mathcal{M} \models d_p\varphi(\overline{b})$.

Next suppose $\varphi(\overline{x}, \overline{b}) \notin p$. Then $\theta'_{\overline{b}}(\overline{x}) = \psi(\overline{x}) \wedge \neg\varphi(\overline{x}, \overline{b})$ is in p. Again, by the minimality of k, we have $\mathcal{M} \models \gamma(\varphi, \theta_{\overline{b}}, k-1)$. Suppose that also $\mathcal{M} \models \gamma(\varphi, \theta_{\overline{b}}, k-1)$. Then we can form a height k tree with root $\overline{b}$, the left tree from the witnesses to $\gamma(\varphi, \theta_{\overline{b}}, k-1)$, and the right tree from the witnesses to $\gamma(\varphi, \theta_{\overline{b}}, k-1)$. But that means $\gamma(\varphi, k)$ holds, which contradicts the definition of k. Thus, $\mathcal{M} \not\models \gamma(\varphi, \theta_{\overline{b}}, k-1)$, i.e. $\mathcal{M} \not\models d_p\varphi(\overline{b})$.

(3) $\Rightarrow$ (4):

Let $\Gamma_T(\varphi, \omega)$ hold. Let λ be any cardinal. Let μ be the least cardinal such that $2^\mu > \lambda$. By Theorem 4, we have $\Gamma_T(\varphi, \mu)$ holds. Unpacking the definition of $\Gamma_T(\varphi, \mu)$, there is an $\mathcal{M} \models T$ and a height μ tree $f : 2^{<\mu} \to M^m$ such that for each branch $\tau \in 2^\mu$,

$$\Sigma_\tau = \{\varphi(\overline{x}, f(\tau \upharpoonright \beta)) \mid \beta < \alpha, \tau(\beta) = 0\} \cup \{\neg\varphi(\overline{x}, f(\tau \upharpoonright \beta)) \mid \beta < \alpha, \tau(\beta) = 1\}$$

is satisfiable in $\mathcal{M}$. For any $\tau \in 2^\mu$, we have Σ_τ is a partial n-type over the paramter set

$$B = \{b \in f(\sigma) \mid \sigma \in 2^{<\mu}\}.$$

Furthermore, let $\tau, \tau' \in 2^\mu$ be distinct. Let $\beta < \mu$ be the minimal ordinal such that $\tau(\beta) \neq \tau'(\beta)$. Define $\sigma = \tau \upharpoonright \beta = \tau' \upharpoonright \beta$. We have $\varphi(\overline{x}, f(\sigma))$ is in one of Σ_τ and $\Sigma_{\tau'}$, while $\neg\varphi(\overline{x}, f(\sigma))$ is in the other. Thus, Σ_τ and $\Sigma_{\tau'}$ are inconsistent with each other. If for each $\tau \in 2^\mu$ we choose some complete type p_τ extending Σ_τ, then we get 2^μ distinct, complete types over B.

Now we determine the size of B. Each element of B comes from some tuple of size n in the tree f. Then

$$|B| \leq n \times |2^{<\mu}| = |2^{<\mu}|.$$

We have

$$2^{<\mu} = \bigcup_{\alpha < \mu} 2^\alpha,$$

[1] $\gamma(\varphi, \theta, k)$ can be defined similarly as in proposition 3

so

$$|2^{<\mu}| \leq \sum_{\alpha<\mu} 2^{|\alpha|}.$$

By construction, for all $\alpha < \mu$, we have $2^{|\alpha|} \leq \lambda$. Also, $2^\lambda > \lambda$, we have $\mu \leq \lambda$. Thus,

$$\sum_{\alpha<\mu} 2^{|\alpha|} \leq \sum_{\alpha<\mu} \lambda = \mu \times \lambda \leq \lambda^2 = \lambda.$$

We have shown T has $2^\mu > \lambda$ complete types over B, and $|B| = \lambda$. That means T is λ-unstable. Since λ was arbitrary, T is unstable.

(4) $\Rightarrow$ (1):

Clearly if for each λ there is an $\mathcal{M} \models T$ and $A \subseteq M$ with $|A| = \lambda$ such that there are more than λ many φ-types, then there are more than λ types.

$\square$