1 Prelminary results about uniformizations

Definition. If $A \subseteq X \times Y$, a *uniformization* of A is a subset $A' \subseteq A'$ such that for each $x \in \pi_1[A]$, there is a unique $y \in Y$ with $(x, y) \in A'$. (A' is the graph of a partial function $X \to Y$ with domain $\pi_1[A]$)

Theorem 1. There is a closed set $C \subseteq \omega^\omega \times \omega^\omega$ with no Borel uniformization.

Proof. We can derive this from a pair of $\mathbf{\Pi}^1_1$ sets that are Borel inseparable.
(Picture proof)

Let $A_1, A_2 \subseteq \omega^\omega$ be the complements of a pair of $\mathbf{\Pi}^1_1$, Borel inseparable sets (such as WF, UB). This implies $\omega^\omega = A_1 \cup A_2$ and there is no Borel B with $B \subseteq A_1$ and $\omega^\omega \setminus B \subseteq A_2$. Take closed $C_1, C_2 \subseteq \omega^\omega \times \omega^\omega$ so that $\pi[C_i] = A_i$. Then $C = C_1 \times \{0\} \cup C_2 \times \{1\} \subseteq \omega^\omega \times 2 \cong \omega^\omega$. Now, suppose F is a Borel uniformization of $C \cup D$. Take $\hat{A}_1 = \pi[C_1 \cap F]$ and $\hat{A}_2 = \pi[C_2 \cap F]$. Then $\hat{A}_1, \hat{A}_2$ partition ω^ω and are $\mathbf{\Sigma}^1_1$, hence are Borel, and $\hat{A}_i \subseteq A_i$.

□

Theorem 2. If $C \subseteq 2^\omega \times 2^\omega$ is closed then C has a Borel uniformization.

Proof. Let $C = [T \times S]$ where $T \times S$ is a tree through $2^{<\omega} \times 2^{<\omega}$. Then we have $\pi[C] = [T]$. Let

$$F = \{(x, y) \in C \mid \forall n \in \omega \ \forall \sigma <_{\text{lex}} y \restriction n \ (N_{x \restriction n} \times N_\sigma \cap C = \emptyset)\}.$$

For each $x \in [T]$:

- $|F_x| \leq 1$ because if y, y' are in F_x, would have $y \restriction n \neq y' \restriction n$, and whichever one is smaller would violate the other being in F_x
- $|F_x| \geq 1$ because we can take

$$y = \bigcup_{n \in \omega} \min\{\sigma \in 2^n \mid N_{x \restriction n} \times N_\sigma \cap C \neq \emptyset\}$$

It remains to check that F is Borel. Checking that $N_{x\restriction n} \times N_\sigma \cap C = \emptyset$ is Borel amounts to checking that $T \times S$ has infinitely many nodes above $(x \restriction s, \sigma)$.

Note: This is where this argument fails for closed subsets of $\omega^\omega \times \omega^\omega$ □

Theorem 3. If $A \subseteq \omega^\omega \times \omega^\omega$ is $\mathbf{\Pi}^1_1$ then A admits a $\mathbf{\Pi}^1_1$ uniformization.

Theorem 4. (Larman, 1973) If $B \subseteq 2^\omega \times 2^\omega$ is such that B_x is closed and uncountable for every $x \in 2^\omega$, then there exist $\aleph_1$ many disjoint Borel uniformizations of B.

Theorem 5. There is a closed set $C \subseteq 2^\omega \times 2^\omega$ such that C_x is uncountable for every x, but there do not exist uncountably many disjoint Borel uniformizations whose rank is bounded below ω_1

Note: This shows that the result by Larman is best possible because the constructed set clearly cannot have $\aleph_2$ *many disjoint Borel uniformizations.*

The following is our plan for proof of theorem

(i) For any α, there is a Borel set which cannot be written as a union of fewer than $\text{cov}(\mathbf{K})$ many sets of bounded rank. (Recall: $\text{cov}(\mathbf{K})$ is the least number of meager sets needed to cover a perfect Polish space.)

(ii) Use this to build a pair of coanalytic sets which cannot be separated by $\text{cov}(\mathbf{K})$ many sets of rank bounded below ω_1.

(iii) Let $Q \subseteq 2^\omega$ be the eventually constant sequences. If A is analytic, there exists a closed set $C \subseteq 2^\omega \times 2^\omega$ so that

- If $x \in A$ then C_x is uncountable.
- If $x \notin A$ then $C_x \subseteq Q$.

Apply this to the complements of the sets from part (ii) to get C_0, C_1.

Let $C = \{(x, y^\frown 0) \mid (x, y) \in C_0\} \cup \{(x, y^\frown 1) \mid (x, y) \in C_1\}$.

(Picture of the diagram here)

(iv) Show that this C satisfies the conclusion of the theorem assuming $\text{cov}(\mathbf{K}) > \aleph_1$.

(v) Use an absoluteness argument to get rid of assumption that $\text{cov}(\mathbf{K}) > \aleph_1$.

2 Part (i)

Theorem 6. For any $\alpha < \omega_1$, there exists a Borel set B which cannot be expressed as the union of fewer than $\text{cov}(\mathbf{K})$ many $\mathbf{\Pi}^0_\alpha$ sets.

This is where we will use tagged tree forcing. Define $\mathcal{T}$ to be the set of trees T which contain $\emptyset$, and such that, if $s \in T$, then either all the successors of s are in T or none are. We will let τ denote the usual topology on $\mathcal{T}$, which is generated by basic clopens of the form $N_t = \{T \in \mathcal{T} \mid t \in T\}$ for $t \in \omega^{<\omega}$.

Definition. A *tagged tree* is a pair $p = (S, h)$, where S is a finite tree and $h : S \to \omega_1 \cup \{\infty\}$. We say p is a *β-tagged tree* if its tags come from $\beta \cup \{\infty\}$, and let $\mathbb{P}_\beta$ denote the collection of β-tagged trees.

For $\beta < \omega_1$, define $R_\beta = \{T \in \mathcal{T} \mid \beta \notin \mathrm{ran}(\mathrm{rk}_T)\}$.

Draw: Picture of R_β

In τ, we can check that $\mathrm{rk}_T(s) = \alpha$ is Borel, for any fixed $s \in \omega^{<\omega}$ and $\alpha < \omega_1$. Hence, each R_β is Borel in τ. Define the topology τ_β on R_β as the topology generated by basic open sets

$$N_p^\beta = \{T \in R_\beta \mid p \subseteq (T, \mathrm{rk}_T)\},$$

where $(S, h) \in \mathbb{P}_\beta$ is a β-tagged tree.

Proposition 7. τ_β is a Polish topology extending the topology inherited by τ.

For $A \subseteq \mathcal{T}$, write

$$p \Vdash A$$

if $A \cap N_p^\beta$ is comeager in N_p^β with the β topology.

Lemma 8. (Retagging Lemma): suppose $\alpha < \omega_1$ and A is $\mathbf{\Pi}^0_\alpha$ in τ. Then for $\beta_1, \beta_2 \geq \omega \cdot \alpha$, and $p_1 \in \mathbb{P}_{\beta_1}$, and $p_2 \in \mathbb{P}_{\beta_2}$, such that $p_1 \sim_{\omega\cdot\alpha} p_2$, we have

$$p_1 \Vdash_{\beta_1} A \Leftrightarrow p_2 \Vdash_{\beta_2} A.$$

Now, we are ready to prove the theorem.

proof of Theorem 6. Let root_∞ be the tagged tree $(\{\emptyset\}, \emptyset \mapsto \infty)$.
Claim: If $\beta' < \beta < \omega_1$ then $N^\beta_{\mathrm{root}_\infty} \cap R_{\beta'}$ is closed, nowhere dense in τ_β.

If $T \in N^\beta_{\mathrm{root}_\infty} \setminus R_{\beta'}$, then take some $s \in T$ with $\mathrm{rk}_T(s) = \beta'$, let S be some finite subtree of T containing s, and let $h : S \to \beta \cup \{\infty\}$ be $\mathrm{rk}_T \restriction S$. Then N_p^β is a neighborhood if T disjoint from $R_{\beta'}$. Also, if $p \supseteq \mathrm{root}_\infty$ is any basic open in $N^\beta_{\mathrm{root}_\infty}$, we can extend this to $p' \supseteq p$ where we add some node of rank β'.

Then for any $\beta' < \beta < \omega_1$, we have

$$\mathrm{root}_\infty \Vdash_\beta R_\beta \setminus R_{\beta'}.$$

Claim: If $\beta \geq \omega \cdot \alpha$ then R_β cannot be written as the union of fewer than $\mathrm{cov}(\mathbf{K})$ many $\mathbf{\Pi}^0_\alpha$ sets (in τ). Suppose R_β were able to be written as a union of fewer than $\mathrm{cov}(\mathbf{K})$ many $\mathbf{\Pi}^0_\alpha$ sets. Then $N^\beta_{\mathrm{root}_\infty}$ is covered by these sets. Regarded as a Polish subspace, it follows by definition that one such set W must be nonmeager. That, is $\mathrm{root}_\infty \not\Vdash_\beta \neg W$ for some $\mathbf{\Pi}^0_\alpha$ set. Take $q \supseteq \mathrm{root}_\infty$ so that $q \Vdash_\beta W$. By the retagging lemma, we also have $q \Vdash_{\beta+1} W$. But also $q \Vdash_{\beta+1} \neg R_\beta$. These are incompatible. □

3 Part (ii)

Lemma 9. There exists a pair of disjoint coanalytic sets C, D which cannot be separated by a union of $\mathrm{cov}(\mathbf{K})$ many Borel sets of rank bounded below ω_1.

Proof. Take U to be universal coanalytic. Define $\tilde{C} = \{(x, y, z) \mid (x, z) \in U\}$ and $\tilde{D} = \{(x, y, z) \mid (y, z) \in U\}$. Apply the reduction principle to get $C \subseteq \tilde{C}$ and $D \subseteq \tilde{D}$ which are disjoint and $C \cup D = \tilde{C} \cup \tilde{D}$

We observe that separators for C, D can be used to express arbitrary Borel sets as follows. If B is Borel, then B and $\omega^\omega \setminus B$ are both coanalytic, so we may find x_0, y_0 so that $B = U_{x_0}$ and $\omega^\omega \setminus B = U_{y_0}$. In particular, we must have $C_{x_0,y_0} = \tilde{C}_{x_0,y_0} = B$, because $\tilde{D}$ contains no points of the form (x_0, y_0, z), for $z \in B$. Therefore, if we have a separator S for C, D expressed as the union of $\mathrm{cov}(\mathbf{K})$ many $\mathbf{\Pi}^0_\alpha$ sets, we may take the (x_0, y_0) sections to express B as the union of $\mathrm{cov}(\mathbf{K})$ many $\mathbf{\Pi}^0_\alpha$ sets. Then the lemma follows from Theorem 6.

□

4 Part (iii)

Lemma 10. Let $Q \subseteq 2^\omega$ be the eventually constant sequences. If A is analytic, there exists a closed set $C \subseteq 2^\omega \times 2^\omega$ so that

- If $x \in A$ then C_x is uncountable.
- If $x \notin A$ then $C_x \subseteq Q$.

Proof. Find $B \subseteq 2^\omega \times \omega^\omega$ closed so that $A = \pi[B]$. Let $B' = B \times \omega^\omega$.

- If $x \in A$ then $B'_x = B_x \times \omega^\omega$ is uncountable.
- If $x \notin A$ then B'_x is empty.

But $\omega^\omega \times \omega^\omega \cong \omega^\omega \cong 2^\omega \setminus Q$. Let f be a homeomorphism $\omega^\omega \times \omega^\omega \to 2^\omega \setminus Q$. Then define

$$C = \mathrm{cl}_{2^\omega \times 2^\omega}(\{(x, f(y,z)) \mid (x,y,z) \in B'\}).$$

For $x \in A$, C_x remains is uncountable since f is a bijection

Since f is a homeomorphism, and B' is closed we have $\{(x, f(y,z)) \mid (x,y,z) \in B'\}$ is closed in $2^\omega \times (2^\omega \setminus Q)$. Hence, taking the closure in $2^\omega \times 2^\omega$ can only add points in $2^\omega \times Q$. Hence, for $x \notin A$, $C_x \subseteq Q$ □

5 Part (iv)

Lemma 11. Theorem 5 holds assuming that $\mathrm{cov}(\mathbf{K}) > \aleph_1$.

Proof. Assume $\mathrm{cov}(\mathbf{K}) > \aleph_1$. Let D_0, D_1 be the disjoint coanalytic sets from part(ii) which we proved cannot be separated by the union of $\mathrm{cov}(\mathbf{K})$ many Borel sets of bounded rank. For each i, let A_i be the complement of D_i. Then $A_0 \cup A_1 = 2^\omega$. Apply lemma 10 to each A_i to get closed sets $C_i \subseteq 2^\omega \times 2^\omega$ such that

- If $x \in A_i$ then $(C_i)_x$ is uncountable.
- If $x \notin A$ then $(C_i)_x \subseteq Q$.

Combine the C_i together into the set

$$C = \{(x, 0^\frown y) \mid (x,y) \in C_0\} \cup \{(x, 1^\frown y) \mid (x,y) \in C_1\}$$

Same picture as from before!

Since $A_0 \cup A_1 = 2^\omega$, we get C_x is uncountable for all x. Suppose to the contrary that there were a sequence $(u_\gamma)_{\gamma<\omega_1}$ of pairwise disjoint Borel functions uniformizing C whose ranks are all bounded below some $\alpha < \omega_1$. For each $\gamma < \omega_1$, let

$$E_\gamma = \{x \in 2^\omega \mid u_\gamma(x) \text{ starts with 1 and } u_\gamma(x) \notin Q\}.$$

Checking that something starts with 1 and is not in Q is Π^0_2, so the rank of E_γ is at most 2 more than the rank of u_γ.

Claim: $\bigcup_{\gamma<\omega_1} E_\gamma$ separates D_0 from D_1.

On the one hand, for any x and γ,

$$x \in D_0 \Rightarrow x \notin A_1 \Rightarrow x \in u_\gamma(x) \text{ starts with } 0 \Rightarrow x \notin E_\gamma.$$

On the other hand, for any x and γ,

$$x \in D_1 \Rightarrow x \notin A_2 \Rightarrow u_\gamma(x) \text{ starts with } 1.$$

For an arbitrary γ, we cannot guarantee that $u_\gamma(x) \notin Q$. However, since u_γ are disjoint uniformizations, there can only be $|Q| = \aleph_0$ many γ for which $u_\gamma(x) \in Q$. Hence, for almost all $\gamma < \omega_1$ we have $x \in E_\gamma$.

This contradicts part (i), since we've just shown that $\bigcup_{\gamma<\omega_1} E_\gamma$ is a union of $\aleph_1 < \mathrm{cov}(\mathbf{K})$ many sets of rank bounded by $\alpha + 2$ which separates D_0 and D_1. □

6 Part (v)

Like any other cardinal characteristic of the continuum, $\mathrm{cov}(\mathbf{K})$ is equal to the continuum under MA. Then in a generic extension of V where ZFC + MA + ¬CH holds, we have in which case $\mathrm{cov}(\mathbf{K}) = 2^{\aleph_0} > \aleph_1$. But Borel codes B for the functions u_γ are absolute between generic extensions of ZFC, and the statements made in Theorem 5 about u_γ are Π^1_2:

- u_γ is a function
- (u_γ) are pairwise disjoint
- $u_\gamma \subseteq B$

By Shoenfield absoluteness, these are absolute. Hence, Theorem 5 holds absolutely.